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One-Parameter Indecomposable Representation of SPL(2,1) Superalgebra

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Using inhomogeneous boson–fermion realization one-parameter indecomposable representation of the SPL(2,1) superalgebra is studied on subspace and quotient spaces of the universal enveloping algebra of Heisenberg-Weyl superalgebra.

KEY WORDS: SPL(2,1) superalgebra; inhomogeneous differential realization; boson–fermion realization; indecomposable representation.

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1. INTRODUCTION

Lie superalgebras have played an important role in nuclear physics, superunification, and supergravity (Balantekin and Bars, 1982). The indecomposable representations of Lie superalgebras have many important applications in description of unstable particle systems (Dirac, 1984). Some indecomposable representations of Lie superalgebras SPL(2,1) and GL(2|1) have been given by Chen (1993, 2001a,b). Therefore, it is very important to study further the new one-parameter indecomposable representation of the SPL(2,1) superalgebra. It is quite a valid approach to employ the inhomogeneous boson–fermion realization of Lie superalgebras in order to study their indecomposable representations. In the present paper, we shall study one-parameter indecomposable representation of the SPL(2,1) superalgebra on the universal enveloping algebra of Heisenberg-Weyl superalgebras, and on their subspaces and quotient spaces using one-parameter inhomogeneous boson–fermion realization of this superalgebra.

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2. ONE-PARAMETER INDECOMPOSABLE REPRESENTATION OF THE SPL(2,1)

In accordance with Chen (1993), the generators of the SPL(2,1) superalgebra read as follows:

$$\{Q_3, Q_+, Q_-, B \in \text{SPL}(2,1)_{\bar{0}} \,|\, V_+, V_-, W_+, W_- \in \text{SPL}(2,1)_{\bar{1}}\} \tag{1}$$

and satisfy the following commutation and anticommutation relations:

$$\begin{aligned}
[Q_3, Q_\pm] &= \pm Q_\pm,\ [Q_+, Q_-] = 2Q_3, [B, Q_\pm] = [B, Q_3] = 0\\
[Q_3, V_\pm] &= \pm\tfrac{1}{2}V_\pm, [Q_3, W_\pm] = \pm\tfrac{1}{2}W_\pm, [BV_\pm,] = \tfrac{1}{2}V_\pm\\
[B, W_\pm] &= -\tfrac{1}{2}W_\pm, [Q_\pm, V_\mp] = V_\pm, [Q_\pm, W_\mp] = W_\pm[Q_\pm, V_\pm] = 0\\
[Q_\pm, W_\pm] &= 0, \{V_\pm, V_\pm\} = \{V_\pm, V_\mp\} = \{W_\pm, W_\pm\} = \{W_\pm, W_\mp\} = 0,\\
\{V_\pm, W_\pm\} &= \pm Q_\pm, \{V_\pm, W_\mp\} = -Q_3 \pm B.
\end{aligned} \tag{2}$$

In terms of one pair of boson operators and two pairs of fermion operators, one-parameter inhomogeneous boson–fermion realization of the SPL(2,1) may be represented as follows (Chen, 2003):

$$\begin{aligned}
Q_3 &= -\frac{1}{2}n + b^+b + \frac{1}{2}a_1^+a_1 + \frac{1}{2}a_2^+a_2,\\
Q_+ &= nb^+ - b^{+2}b - b^+a_1^+a_1 - b^+a_2^+a_2,\\
Q_- &= b,\\
B &= \left(\frac{1}{2} + \alpha\right)n - \frac{1}{2}a_1^+a_1 - \frac{1}{2}a_2^+a_2,\\
V_+ &= \sqrt{1+\alpha}na_2^+ + \sqrt{\alpha}b^+a_1 - \sqrt{1+\alpha}a_2^+b^+b - \sqrt{1+\alpha}a_2^+a_1^+a_1,\\
V_- &= -\sqrt{1+\alpha}a_2^+b + \sqrt{\alpha}a_1,\\
W_+ &= -\sqrt{\alpha}na_1^+ + \sqrt{1+\alpha}b^+a_2 + \sqrt{\alpha}a_1^+b^+b + \sqrt{\alpha}a_1^+a_2^+a_2,\\
W_- &= \sqrt{\alpha}a_1^+b + \sqrt{1+\alpha}a_2.
\end{aligned} \tag{3}$$

Consider (1+2) states Heisenberg–Weyl superalgebra

$$H : b^+, b, a_1^+, a_1, a_2^+, a_2, E$$

where E stands for the unit operator. According to the Poincare–Birckhoff–Witt theorem, we choose for its universal enveloping algebra Ω a basis

$$\begin{aligned}
\{\phi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2, t) = b^{+k}b^l a_1^{+\alpha_1}a_1^{\beta_1}a_2^{+\alpha_2}a_2^{\beta_2}E^t | k, l, t \in Z^+,\\
\alpha_1, \beta_1, \alpha_2, \beta_2 = 0, 1\}
\end{aligned} \tag{4}$$

Each vector in the space of Ω is a linear combination of the basis with complex coefficients. Then, we consider an extension $\bar{\Omega}$ of the space Ω, in which each element is a linear combination of the basis whose coefficients are elements of the Grassmann algebra $\tilde{G}$.

The representation of the superalgebra H on the space of $\bar{\Omega}$ is defined as

$$
\begin{aligned}
f(b^+)\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2,t) &= \phi(k+1,l,\alpha_1,\beta_1,\alpha_2,\beta_2,t)\\
f(b)\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2,t) &= \phi(k,l+1,\alpha_1,\beta_1,\alpha_2,\beta_2,t)\\
&\quad + k\phi(k-1,l,\alpha_1,\beta_1,\alpha_2,\beta_2,t+1)\\
f(a_1^+)\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2,t) &= (1-\alpha_1)\phi(k,l,\alpha_1+1,\beta_1,\alpha_2,\beta_2,t)\\
f(a_1)\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2,t) &= (-1)^{\alpha_1}\phi(k,l,\alpha_1,\beta_1+1,\alpha_2,\beta_2,t)\\
&\quad + \alpha_1\phi(k,l,\alpha_1-1,\beta_1,\alpha_2,\beta_2,t+1)\\
f(a_2^+)\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2,t) &= (-1)^{\alpha_1+\beta_1}(1-\alpha_2)\times\phi(k,l,\alpha_1,\beta_1,\alpha_2\\
&\quad +1,\beta_2,t)\\
f(a_2)\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2,t) &= (-1)^{\alpha_1+\beta_1+\alpha_2}\times\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2+1,t)\\
&\quad +(-1)^{\alpha_1+\beta_1}\alpha_2\phi(k,l,\alpha_1,\beta_1,\alpha_2-1,\beta_2,t+1)
\end{aligned} \tag{5}
$$

Now, we consider the quotient space V with the basis

$$
\begin{aligned}
V &= (\bar{\Omega}/I) : \{\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&= \phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2,0) \bmod I \,|\, k,l \in Z^+, \alpha_1,\beta_1,\alpha_2,\beta_2 = 0,1\}
\end{aligned} \tag{6}
$$

corresponding to the two-sided ideal I generated by the element $E = 1$.

The representation (5) induces the new representation on the space of V

$$
\begin{aligned}
f(b^+)\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2) &= \phi(k+1,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
f(b)\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2) &= \phi(k,l+1,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&\quad + k\phi(k-1,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
f(a_1^+)\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2) &= (1-\alpha_1)\phi(k,l,\alpha_1+1,\beta_1,\alpha_2,\beta_2)\\
f(a_1)\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2) &= (-1)^{\alpha_1}\phi(k,l,\alpha_1,\beta_1+1,\alpha_2,\beta_2)\\
&\quad + \alpha_1\phi(k,l,\alpha_1-1,\beta_1,\alpha_2,\beta_2)\\
f(a_2^+)\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2) &= (-1)^{\alpha_1+\beta_1}(1-\alpha_2)\times\phi(k,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2)\\
f(a_2)\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2) &= (-1)^{\alpha_1+\beta_1+\alpha_2}\times\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2+1)\\
&\quad + (-1)^{\alpha_1+\beta_1}\alpha_2\phi(k,l,\alpha_1,\beta_1,\alpha_2-1,\beta_2)
\end{aligned} \tag{7}
$$

Using the following relation

$$
L(F(b^+,b,a_1^+,a_1,a_2^+,a_2)) = \tilde{F}(f(b^+),f(b),f(a_1^+),f(a_1),f(a_2^+),f(a_2)) \tag{8}
$$

and the one-parameter inhomogeneous boson–fermion realization (3), we obtain the representation L of the SPL(2,1) on the space of V,

$$\begin{aligned}
L(Q_3)\ &\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\left(-\tfrac{1}{2}n+k+\tfrac{1}{2}\alpha_1+\tfrac{1}{2}\alpha_2\right)\\
&\times\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)+\phi(k+1,l+1,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&+\tfrac{1}{2}(-1)^{\alpha_1}(1-\alpha_1)\,\phi(k,l,\alpha_1+1,\beta_1+1,\alpha_2,\beta_2)\\
&+\tfrac{1}{2}(-1)^{\alpha_2}(1-\alpha_2)\phi(k,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2+1)\\
L(Q_+)\ &\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)=(n-k-\alpha_1-\alpha_2)\\
&\times\phi(k+1,l,\alpha_1,\beta_1,\alpha_2,\beta_2)-\phi(k+2,l+1,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&-(-1)^{\alpha_1}(1-\alpha_1)\phi(k+1,l,\alpha_1+1,\beta_1+1,\alpha_2,\beta_2)\\
&-(-1)^{\alpha_2}(1-\alpha_2)\phi(k+1,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2+1)\\
L(Q_-)\ &\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)=\phi(k,l+1,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&+k\phi(k-1,l,\alpha_1,\beta_1,\alpha_2,\beta_2)
\end{aligned}\tag{9}$$

$$\begin{aligned}
L(B)\ &\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)=\left[\left(\tfrac{1}{2}+\alpha\right)n-\tfrac{1}{2}\alpha_1-\tfrac{1}{2}\alpha_2\right]\\
&\times\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\\
&+\tfrac{1}{2}(-1)^{\alpha_1}(1-\alpha_1)\phi(k,l,\alpha_1+1,\beta_1+1,\alpha_2,\beta_2)\\
&\tfrac{1}{2}(-1)^{\alpha_2}(1-\alpha_2)\phi(k,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2+1)
\end{aligned}$$

$$\begin{aligned}
L(V_+)\ &\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)=(-1)^{\alpha_1+\beta_1}(1-\alpha_2)(n-k-\alpha_1)\\
&\times\sqrt{1+\alpha}\,\phi(k,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2)\\
&+(-1)^{\alpha_1}\sqrt{\alpha}\,\phi(k+1,l,\alpha_1,\beta_1+1,\alpha_2,\beta_2)\\
&+\alpha_1\sqrt{\alpha}\,\phi(k+1,l,\alpha_1-1,\beta_1,\alpha_2,\beta_2)\\
&-(-1)^{\alpha_1+\beta_1}(1-\alpha_2)\sqrt{1+\alpha}\,\phi(k+1,l+1,\alpha_1,\beta_1,\alpha_2+1,\beta_2)\\
&-(-1)^{\beta_1}(1-\alpha_1)(1-\alpha_2)\sqrt{1+\alpha}\,\phi(k,l,\alpha_1+1,\beta_1+1,\alpha_2+1,\beta_2)\\
L(V_-)\ &\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)=(-1)^{\alpha_1}\sqrt{\alpha}\,\phi(k,l,\alpha_1,\beta_1+1,\alpha_2,\beta_2)\\
&+\alpha_1\sqrt{\alpha}\,\phi(k,l,\alpha_1-1,\beta_1,\alpha_2,\beta_2)\\
&(-1)^{\alpha_1+\beta_1}(1-\alpha_2)\sqrt{1+\alpha}\,\phi(k,l+1,\alpha_1,\beta_1,\alpha_2+1,\beta_2)\\
&(-1)^{\alpha_1+\beta_1}(1-\alpha_2)\sqrt{1+\alpha}k\,\phi(k-1,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2)\\
L(W_+)\ &\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)=(-n+k+\alpha_2)(1-\alpha_1)\sqrt{\alpha}\\
&\times\phi(k,l,\alpha_1+1,\beta_1,\alpha_2,\beta_2)\\
&+(-1)^{\alpha_1+\beta_1+\alpha_2}\sqrt{1+\alpha}\,\phi(k+1,l,\alpha_1,\beta_1,\alpha_2,\beta_2+1)\\
&+(-1)^{\alpha_1+\beta_1}\alpha_2\sqrt{1+\alpha}\,\phi(k+1,l,\alpha_1,\beta_1,\alpha_2-1,\beta_2)\\
&+(1-\alpha_1)\sqrt{\alpha}\,\phi(k+1,l+1,\alpha_1+1,\beta_1,\alpha_2,\beta_2)\\
&-(-1)^{\alpha_2}(1-\alpha_1)(1-\alpha_2)\sqrt{\alpha}\,\phi(k,l,\alpha_1+1,\beta_1,\alpha_2+1,\beta_2+1)\\
L(W_-)\ &\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)=(-1)^{\alpha_1+\beta_1+\alpha_2}\sqrt{1+\alpha}\\
&\times\phi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2+1)\\
&+(-1)^{\alpha_1+\beta_1}\alpha_2\sqrt{1+\alpha}\,\phi(k,l,\alpha_1,\beta_1,\alpha_2-1,\beta_2)\\
&+(1-\alpha_1)\sqrt{\alpha}\,\phi(k,l+1,\alpha_1+1,\beta_1,\alpha_2,\beta_2)\\
&-(1-\alpha_1)\sqrt{\alpha}\,k\phi(k-1,l,\alpha_1+1,\beta_1,\alpha_2,\beta_2)
\end{aligned}$$

From (9), it follows that the sum $(l + \beta_1 + \beta_2)$ does not decrease under the action of the representation L and the subspace

$$V_m = \{\phi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2) \in V \mid l + \beta_1 + \beta_2 m\}$$

is invariant, for which no invariant complementary subspace exists. Thus, the representation given by (9) on the space V is indecomposable.

3. CONCLUSION

We have obtained one-parameter indecomposable representation of the SPL(2,1) superalgebra on the universal enveloping algebra of Heisenberg–Weyl superalgebras, and on their subspaces and quotient spaces using one-parameter inhomogeneous boson–fermion realization of this superalgebra. In terms of the conclusion it may be of use for further research on one-parameter irreducible representations of the SPL(2,1) superalgebra.

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